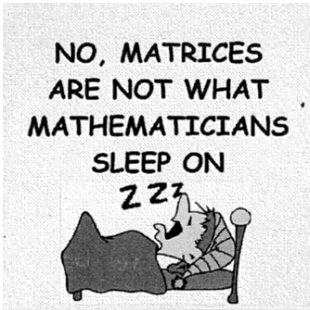
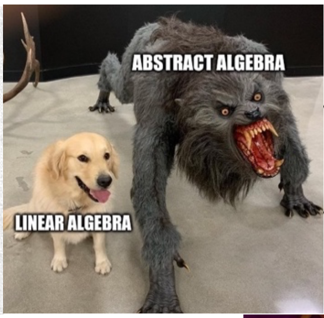
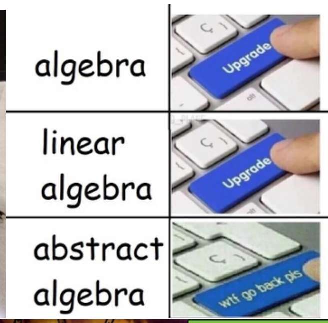
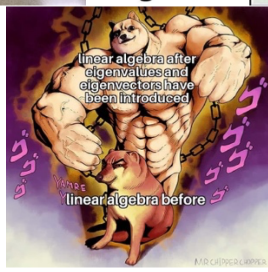
Linear Algebra: Types Of Functions, Isomorphisms and Homomorphisms









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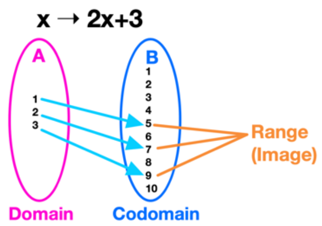
1 Types Of Functions

In its simplest form the domain is all the values that go into a function (x values), and the range is all the values that come out (y values). The domain and range are so important in defining a function.

There are special names for what can go into, and what can come out of a function

- What can go into a function is called the Domain
- What may possibly come out of a function is called the Codomain
- What actually comes out of a function is called the Range



Set A is called the Domain
Set B is the Codomain
The set of values in B that get pointed to (the actual values produced by the function) are the Range, also called the image
So, we have

- Domain {1,2,3,}
- Codomain {1,2,3,4,5,6,7,8,9,10}
- Range {5,7,9}

Notation:
f: Domain $\longmapsto$ Codomain
$f: x \longmapsto y$ says function f takes x and returns x^2

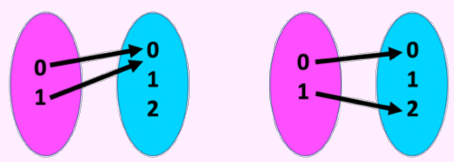
You need to know the definitions of the following 3 types of functions:

- **Injective** is one to one (no 2 or more $x's$ go to the same y i.e. no 2 different x values will give the same y value)
- **Surjective** is onto (everything in the range is hit/touched so codomain equals the range)
- **Bijective** means injective and surjective (invertible linear map)

Let's take a look at these functions in more details.

1.1.1 Injective (one-to-one)

Every element in codomain is mapped to by at least 1 element of its domain. Consider a set A and a set B. We cannot have two or more "A's" pointing to the same "B". We can have a "B" without a matching "A" though. The function must pass horizontal line and vertical line test to be injective. This function is one-to-one, not many to one!



The diagram on the left is not injective. You can't go to the same number (the function doesn't pass horizontal line test here)

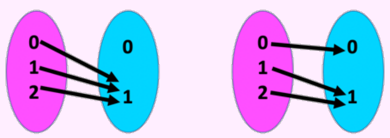
The diagram on the right is injective

Formally we say $f: X \longrightarrow Y$ is injective if and only if $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$

Method to prove injective:
Way 1: Replace function with x_1 and function with x_2 and set equal and see if you get $x_1 = x_2$ only
Way 2: Graph and show passes vertical and horizontal line test

1.1.2 Surjective (onto)

Range/Image is equal to codomain. Consider a set A and a set B. Every "B" has at least one matching "A" i.e. there won't be a "B" left out.



The diagram on the left is not surjective. 0 isn't "touched"

The diagram on the right is surjective

Formally we say $f: X \longrightarrow Y$ is surjective iff $\forall y \in Y, \exists x \in X \; s.t. f(x) = y$
(for every y in the codomain y there is at least one x in the domain such that $f(x) = y$)

Method to prove surjective:
Set $f(x) = y$ and rearrange for x. Show $x \in X$

1.1.3 Bijective

Must be injective and surjective. Think of a perfect pairing between the sets. Everyone has a partner and no one is left out.
Note: A function has an inverse if and only if it is injective and surjective i.e. bijective.

Method to prove bijective:
Prove injective and surjective.

$f: \mathbb{Z} \longrightarrow \mathbb{Z} : f(x) = 3x + 2$

Injective:
$Let \; a_1, a_2 \in \mathbb{Z} \; s.t. f(a_1) = f(a_2) \Leftrightarrow 3a_1 + 2 = 3a_2 + 2$
$\Rightarrow 3\, a_1 = 3a_2$
$\Rightarrow a_1 = a_2$
∴ injective

Surjective:
$f(x) = 3x + 2 = y$
$y = 3x + 2$
$x = \frac{y-2}{3}$
x is sometimes an integer and sometimes not an integer, dependent on what y is
take $y = 0$ as a counterexample $\Rightarrow x \notin \mathbb{Z}$
∴ not surjective

Injective, but not surjective ∴ not bijective

$f: \mathbb{Q} \longrightarrow \mathbb{Q} : f(x) = 3x + 2$

Injective:
$Let \; a_1, a_2 \in \mathbb{Q} \; s.t. f(a_1) = f(a_2) \Leftrightarrow 3a_1 + 2 = 3a_2 + 2$
$\Rightarrow 3\, a_1 = 3a_2$
$\Rightarrow a_1 = a_2$
∴ injective

Surjective:
$f(x) = 3x + 2 = y$
$y = 3x + 2$
$x = \frac{y-2}{3}$
$\Rightarrow x \in \mathbb{Q}, y \in \mathbb{Q}$
∴ surjective

Bijective since both injective and surjective

2 Isomorphisms and Homomorphisms

2.1 Isomorphisms

A tip to get a good feel for what isomorphisms mean, and why we think of isomorphic structures as the same is to learn basic group theory (and their isomorphisms). Finite groups are more concrete than vector spaces (over R), so it will give you a better feel.

A few scenarios to get some intuition:

- Suppose that $\mathbb{R}^3$ throws a masquerade ball. Everyone puts on a costume, so (2,3,5) looks like 2+3x+5x^2. Everyone looks different, but secretly everything is the same. It's still the same people and the same relationships. Previously we would say that $(2,3,5)+(1,2,3)=(3,5,8)$Now, dressed up in costumes, we say that 2+3x+5x^2+1+2x+3$x^2 = 3+5\text{x}+8x^2$. But once you know how to take off the costumes, you see that nothing has changed. An isomorphism tells you how to take the masks off, revealing that everything is the same.
- playing checkers with red and black disks vs. playing checkers with 10 cent and 25 cent coins. Or playing tic-tac-toe with O's and X's vs. playing tick-tack-toe with A's and B's.
- Integer arithmetic. If humans do it, they typically write the integers in the form of decimal digit strings with an optional sign in front; those digits themselves being patterns drawn on a surface like paper. When computers do it, they represent the integers in binary, and the digits are really different charge states of capacitors somewhere in the computer. Now, decimal digits are something different than binary digits, and patterns drawn on paper definitely are something very different than charge states of capacitors. And yet both the human and the computer will come to the result that multiplying 6 by 7 gives 42. That is, although the differences are vast, they are not relevant for the question of arithmetic (they are of course relevant for other questions, for example if the result will survive a power outage). That is, as far as arithmetic goes, those capacitor states are isomorphic to the patterns drawn on paper. The same is true for isomorphic vector spaces. As long as you only care about their vector space properties, you don't need to care about whether you have pairs of real numbers, a single complex number, a real function of the form $x \to ax+b$, a translation in the Euclidean plane, or whatever other isomorphic vector space you have. You will always get the very same results. For example, you will in all cases alike find that you need exactly two basis vectors to span the whole space. And importantly, if you figure out any property in one of the spaces, and it is a property that only refers to the vector space structure, then you will immediately know that it will be exactly the same in all the other isomorphic vector spaces. Just like in the arithmetic example, knowing that in the computer's capacitor-charge representation 6×7=42 means that you also know that if you use the symbols-on-paper representation to work it out, you'll come to the exact same result. Even though in the computer, the 42 will be represented by the binary digit string 101010 (or a corresponding pattern of three charged and three uncharged capacitors), and on your paper the same number will be represented by a pattern of lines representing the digit 4 followed by the digit 2.

Still confused? Consider, $\frac{6}{2}$ and 3. We're concerned with their value, not how they look. We regard them as equal (the same). In this context our notion of sameness is equality. If I draw 2 triangles same shape and size. Say one is in quadrant 2 and the other is in quadrant 4. We regard them as sets of points in a plane, they are different. But in the context of geometry we don't care that they are located in different places. We only care they the same shape and size and we say those are congruent so in the context of geometry our notion of sameness is congruence. So, we have seen above that with geometry the notion of 'sameness' is shapes having same shape and size. What about for a general function? For 2 objects (shapes, set, vector spaces, rings, groups etc) the notion of 'sameness' is those that preserve the properties that you're interested in at the time.

Consider a function from A to B $f: A \to B$. This function f preserves important properties and is bijective (has an inverse and can go back the other way). We also want an inverse to preserve those properties. When this happens, we regard A and B as being the same (isomorphic). We say A$\cong B$. Consider an intuitive example. Let's say you're playing chess and halfway through you have to leave to go to your friend's house. You make a note where all the pieces are and all the information about the game. Later on when at your friend's house you take out a Mario themed chess set and continue the game. You can still continue the game without any problems. The only difference between the 2 chess sets is the superficial difference and has nothing to do with the actual game of chess. It is outside the scope of the game, so you can continue the game no problem. Just plug the pieces into the isomorphism that preserves the structure of the game i.e. f(original chess piece)=mario chess piece. If, however, you took out a monopoly board and wanted to finish chess game, you could not. They are completely different. They don't just differ in a superficial way, they have different properties. Therefore these 2 games are not isomorphic.

It should now make sense that Isomorphic to something else means that they have the same vector structure. Isomorphic spaces are treated the same since "isomorphic" literally means "same structure". An isomorphism is a bijection with special conditions between the operations in the two different spaces. This basically means that if two spaces are isomorphic, their structure will be the same because the operations work in the same way. In other words, two isomorphic spaces are two different representations of the same structure.

Given two objects A and B (which are of the same type; maybe groups, or rings, or vector spaces... etc.), an isomorphism from A to B is a bijection $f: A \to B$ which, in some sense, respects the structure of the objects. In other words, they basically identify the two objects as actually being the same object, after renaming of the elements. Two vector spaces V and W over some field $\mathbb{F}$ are said to be isomorphic if there exists a bijection (invertible linear map) between them i.e. linear maps which are bijections are called vector space isomorphisms, or just isomorphisms. Notice that the existence of a linear map between two vector spaces requires that they have the same field of scalars. Formally we write: a linear transformation T : V → W is called an isomorphism if it is both onto and one-to-one (and hence is a bijection) and we say V and W are isomorphic and write V ≅ W when this is the case. In other words, two vector spaces V and W over the same field F are isomorphic if there is a bijection T: V → W which preserves addition and scalar multiplication, that is, for all vectors u and v in V, and all scalars c∈F,

$$\mathrm{T}(u+v)=\mathrm{T}(u)+\mathrm{T}(v) \text{ and } \mathrm{T}(cv)=c\mathrm{T}(v) \text{ where } T(u+v)\in W \text{ and } T(u),T(v)\in W$$

The correspondence T is called an isomorphism of vector spaces. Two finite-dimensional vector spaces, V and W, are isomorphic to each other if and only if they have the same dimension. A vector space isomorphism is also an equivalence relation (see chapter 3).

Let's use groups to clarify in a more formal way what we mean when we talk about "structure" (do not worry about this if you haven't covered groups yet). We are speaking about a specific operation that identifies the structure in the single case. For example, let (A,*) and (B,⊠) be 2 groups with their respective operations * and ⊠. Then they are isomorphic is there exists a bijection $f: A \to B$ such that $f(a_1 * a_2) = f(a_1) \boxtimes f(a_2)$ where $f(a_1), f(a_2) \in B$. Take note that f must be bijective **and** $f(identity\ A) = Identity\ B$. These 2 groups are isomorphic $(A \cong B)$ which as already mentioned means "equal form" i.e. a notion of "sameness".

2.2 Homomorphisms

Homomorphisms are any functions that preserve the algebraic structure, whereas isomorphisms are bijections that preserve the algebraic structure. This means homomorphisms preserve the algebraic structure without requiring bijection. In the case of vector spaces, the term 'linear transformation; is used in preference to 'homomorphism'. In any case of a vector space, a homomorphism is just a linear transformation.
There are multiple types of homomorphism. There are homomorphisms of groups, rings, modules, algebras and more. More generally there is the concept of morphism in general, which is broadly speaking, a structure preserving a map between mathematical structures. You will most often come across homomorphisms when you first study groups. A **homomorphism** preserves the group structure in each group. It is a tool for comparing two groups for similarities. Sometimes two groups are more than similar, they are identical. Suppose we have two groups we want to compare. They can be any type of group (finite, infinite, commutative, non-commutative)
(G,*) and (H,⊠)
$f: G \to H$ such that $f(x * y) = f(x) \boxtimes f(y)$. We call f a (group) homomorphism.
Example:
$f: \mathbb{Z} \to \mathbb{Z}$. Group operation: +
$f(x) = 2x$
We need to check whether $f(x + y) = f(x) + f(y)$? Does $2(x + y) = 2x + 2y$. Yes! So, f is a homomorphism.
After groups, homomorphisms are the most important concept in abstract algebra. It allows us to connect similar and identical groups. They are also an essential tool for identifying the fundamental building blocks of groups. As we move forward through abstract algebra this idea will recur again and again and again. New objects such as fields, rings, vector spaces will be introduced and then define homomorphisms between them.

An **isomorphism** is a homomorphism, but more. An isomorphism is a homomorphism that is onto and one to one and says our objects of interest are identical. If you want to know more about homomorphisms, then any introductory book to topology is a great place to dive into them. "Introduction to Topology" by Bert Mendelson and "Topology: A first course by Munkres" are great starting textbooks.

3 Equivalence Relations/Equivalence Classes

Let's get some intuition into this first. Let us declare that two television sets are "similar" if they have the same screen size. This is called a **relation**. Two TV sets can be similar, or they can be not similar. A relation is just this kind of thing: any two objects may be related, or not. "Greater than", "equal to", "divides", "is friends with", "loves" are all relations. The "similar" relation for TV sets is actually a special kind of relation: it is an equivalence relation. That means that any TV is similar to itself, that if TV 1 is similar to TV 2 then 2 is also similar to 1, and that if 1 is similar to 2 and 2 is similar to 3, then 1 is also similar to 3. All of these conditions are pretty obviously true for the situation where two TV sets are **similar** if they have the same screen size. Not all relations are like that: for instance, "being Facebook friends with someone" isn't, because if A is friends with B and B is friends with C, it's not necessarily the case that A is friends with C. But since our "similar" relation for TVs is an equivalence relation, we can form equivalence classes: those are sets of TVs that are all similar to each other. We make sure not to leave out of an equivalence class any TV which can be added to it. An equivalence class contains **all** the TVs that are similar to any (hence all) of its members. For example, we have the 21" equivalence class. All 21" TV sets are in this class, and all TV sets with other screen sizes are not. We also have a 24" equivalence class, and a 27" class, and a 42" class etc. The classes are disjoint - there's no TV that belongs to two different classes. The classes are also exhaustive: no TV set is missed. If there happens to be a TV set with screen size 4.4" and it's the only one of its kind, it will belong to a class all its own. Everything is in exactly one class - no more, no less. We'll look at more formal definitions below.

Example of relations with formal notations are
$\{x \in \mathbb{Z} | x = x\}$
$\{x, y \in \mathbb{Z} | x \leq y\}$
$\{x, y \in \mathbb{Z} \times \mathbb{Z} || x \text{ divides } y \text{ i.e. } x|y\}$

Formally, an **equivalence class** is the name that we give to the subset of S which includes **all elements that are equivalent to each other**. "Equivalent" is dependent on a specified relationship, called an equivalence relation. If there's an equivalence relation between any two elements, they're called equivalent.
In other words, any items in the set that are equal belong to the defined equivalence class. This set seems like a rather trivial set, but there are other equivalence relations which make things rather more interesting. We'll look at a few of the simpler ones below. An equivalence class might be defined with an equals sign. 'The equivalence class of a consists of the set of all x, such that x = a'. In other words, any items in the set that are equal belong to the defined equivalence class.

Let's represent our equivalence relation by ~ (it may really be =, ≤, <, >, ≥, | or any number of things). The <u>relation</u> ~ is an equivalence relation if and only if:

1. It is **reflexive**: any a in X must always be equivalent to itself; we can write this as a ~ a.
2. It is **symmetric**: Suppose a, b are in X. Then, if a is equivalent to b, b will also be equivalent to a. We can write this as if a ~ b, b ~ a. This means we can swap the order and we get the same thing.
3. It is **transitive**: Let a, b, and c be elements of X. Then, if a is equivalent to b, and b is equivalent to c, a will also be equivalent to c. We can write this as: for a, b, c in X; if $a \sim b$ and $b \sim c$ it follows that $a \sim c$.

Once we've checked to make sure our relation ~ satisfies the three properties above, we can write the definition of an equivalence class of an element a like this. $[a]$ or $[\![a]\!]$ = { x ∈ X | $a \sim x$}. We read this as "the equivalence class of a consists of the set of all x in X such that a and x are related by ~ to each other".

$\{x, y \in \mathbb{Z} \times \mathbb{Z} \| x = y\}$ $\mathbb{Z} = \{\ldots, -3, -2, -2, -1, 0,1,2,3, \ldots.\}$ Let's try this with some numbers Reflexive: $1 = 1$, $2 = 2$, $3 = 3$, etc all true ∴ reflexive Symmetric: $2 = 3 \iff 3 = 2$, $2 = 4 \iff 4 = 2$, $-1 = 2 \iff 2 = -1$ etc all true ∴ symmetric Note: We don't need the original to be a true statement. We just need the symmetry to hold if the original is said to be true. Transitive: $2 = 4$ and $4 = 5 \Longrightarrow 2 = 5$ true∴ transitive The relation $x = y$ IS an equivalence class
$\{x, y \in \mathbb{Z} \times \mathbb{Z} \| x \neq y\}$ $\mathbb{Z} = \{\ldots, -3, -2, -2, -1, 0,1,2,3, \ldots.\}$ Let's try this with some numbers Reflexive: $1 \neq 1$ ∴ not reflexive Symmetric: $2 \neq 3 \Longrightarrow 3 \neq 2$, $2 \neq 4 \Longrightarrow 4 \neq 2$, $4 \neq 4 \Longrightarrow 4 \neq 4$ etc all true ∴ symmetric Note: We don't need the original to be a true statement. We just need the symmetry to hold if the original is said to be true. Transitive: $2 \neq 4$ and $4 \neq 5 \Longrightarrow 2 \neq 5$ true, but $2 \neq 1$ and $1 \neq 2 \Longrightarrow 2 \neq 2$ not true ∴ not transitive The relation $x \neq y$ is NOT an equivalence class (reflexivity and transitivity break down) Note: This is the same as the relation $x - y \neq 0$
$\{x, y \in \mathbb{Z} \times \mathbb{Z} \| x < y\}$ $\mathbb{Z} = \{\ldots, -3, -2, -2, -1, 0,1,2,3, \ldots.\}$ Let's try this with some numbers Reflexive: $1 < 1$ not true ∴ not reflexive Symmetric: $2 < 3 \Longrightarrow 3 < 2$ not true ∴ not symmetric Note: We don't need the original to be a true statement. We just need the symmetry to hold if 1 is said to be true. Transitive: $2 < 4$ and $4 < 5 \Longrightarrow 2 < 5$ true ∴ transitive The relation $x < y$ is NOT an equivalence class (reflexivity and symmetry break down)
$\{x, y \in \mathbb{Z} \times \mathbb{Z} \| x \leq y\}$ $\mathbb{Z} = \{\ldots, -3, -2, -2, -1, 0,1,2,3, \ldots.\}$ Let's try this with some numbers Reflexive: $3 \leq 3, 4 \leq 4$ *etc* true ∴ reflexive Symmetric: $2 \leq 3 \Longrightarrow 3 \leq 2$ not true ∴ not symmetric Note: We don't need the original to be a true statement. We just need the symmetry to hold if 1 is said to be true. Transitive: $1 \leq 2$ and $2 \leq 3 \Longrightarrow 1 \leq 3$ true ∴ transitive

The relation $x \leq y$ is NOT an equivalence class (symmetry breaks down)

$\{(x, y) \in A \times A| \;|y\}$
A= $\{1,2,3,4,5\}$
Let's try this with some numbers
Reflexive: (1,1) means 1|1 which is true, (2,2) means 2|2 which is true etc ∴ reflexive
Symmetric: (1,1) means 1|1 true, (2,1) means 2|1 which is not true ∴ not symmetric
Transitive: 1|2 and 2|4 $\Longrightarrow$ 1|4 true ∴ transitive
The relation $x|y$ is NOT an equivalence class (symmetry breaks down)

$\{(x, y \in S \times S|\sim\}$
$\sim = \{(1,1), (2,2), (3,3), (4,4)(5,5), (1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$
Reflexive: $(1,1), (2,2), (3,3), (4,4)(5,5) \in S$ ∴ reflexive
Symmetric: $(1,2), (2,1) \in S$, $(1,3), (3,1) \in S\ etc$ ∴ symmetric
Transitive: $(1,1)$ and $(1,3) \Longrightarrow (3,1), (2,3)$ and $(3,2) \Longrightarrow (2,2)$ ∴ etc transitive
Now let's look at the actual equivalence classes
$[\![1]\!]$=everything that 1 is related to =\{1,2,3\}
$[\![2]\!]$=everything that 2 is related to =\{2,1,3\}
$[\![3]\!]$=everything that 3 is related to =\{3,2,1\}
$[\![4]\!]$=everything that 4 is related to =\{4\}
$[\![4]\!]$=everything that 4 is related to =\{5\}
$[\![1]\!]$, $[\![2]\!]$= and $[\![3]\!]$ have the same equivalence class
You should see that:

- The equivalence class is always some subset of S
- Different elements can have the same equivalence class
- If 2 elements are related to each other they will have the same equivalence class

These are important theorems that you will prove in your course.